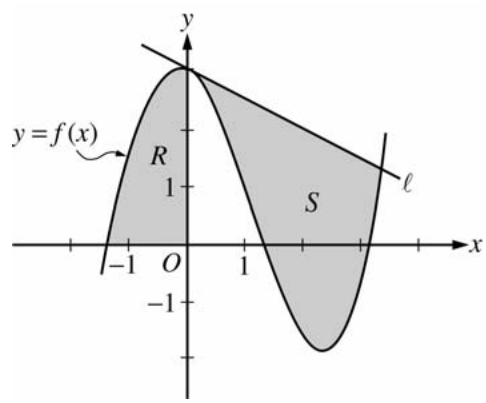
Question 1



Let f be the function given by $f(x) = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x$. Let R be the shaded region in the second quadrant bounded by the graph of f, and let S be the shaded region bounded by the graph of f and line ℓ, the line tangent to the graph of f at $x = 0$, as shown above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(c) Write, but do not evaluate, an integral expression that can be used to find the area of S.

For $x < 0$, $f(x) = 0$ when $x = -1.37312$.
Let $P = -1.37312$.

(a) Area of $R = \int_P^0 f(x)\,dx = 2.903$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_P^0 \left((f(x) + 2)^2 - 4 \right) dx = 59.361$

$4 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) The equation of the tangent line ℓ is $y = 3 - \frac{1}{2}x$.

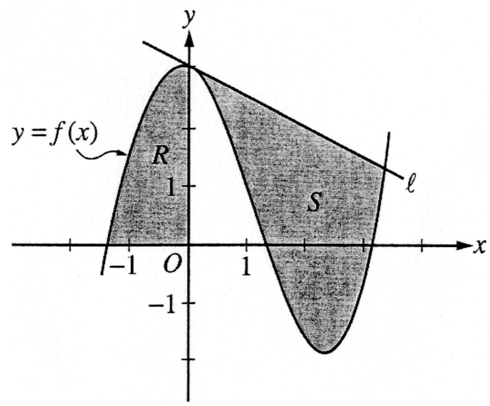
The graph of f and line ℓ intersect at $A = 3.38987$.

Area of $S = \int_0^A \left(\left(3 - \frac{1}{2}x\right) - f(x) \right) dx$

$3 : \begin{cases} 1 : \text{tangent line} \\ 1 : \text{integrand} \\ 1 : \text{limits} \end{cases}$

CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

Area $R = \int_a^b f(x)\,dx$

$f(x) = 0 \Rightarrow \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x = 0$

$x = a = -1.373$

$b = 0$

$R = \int_{-1.373}^{0} f(x)\,dx = 2.903$

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Continue problem 1 on page 5.

Work for problem 1(b)

$$V = \pi \int_{-1.373}^{0} \left[\left(f(x)+2\right)^2 - (2)^2 \right] dx = 59.361 \text{ unit}^3$$

Work for problem 1(c)

$$f'(x) = \frac{3x^2}{4} - \frac{2x}{3} - \frac{1}{2} - 3\sin x$$

$f'(0)$ = slope at $x = 0$

$$f'(0) = -\frac{1}{2}$$

$$f(0) = 3$$

$$y - 3 = -\frac{1}{2}(x - 0)$$

$$y = 3 - \frac{x}{2}$$

intersection point when $3 - \frac{x}{2} = f(x) \Rightarrow x = 3.390$

$$\text{Area } S = \int_0^{3.390} \left[3 - \frac{x}{2} - \left(\frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x \right) \right] dx$$

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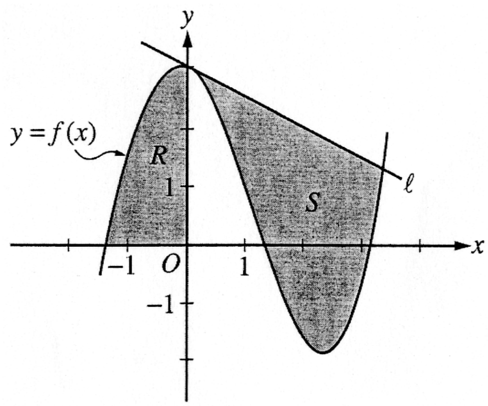
1B

CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

$\frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x = 0$

$x = -1.37312$

$\int_{-1.37312}^{0} \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x \, dx$

$= 2.90309$

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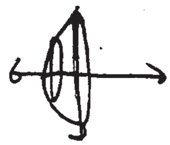
Continue problem 1 on page 5.

Work for problem 1(b)



$r = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x$

$$\int_{-1.37312}^{0} \pi r^2 \, dx = \int_{-1.37312}^{0} \pi \left(\frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x\right)^2 dx = 22.880$$

Work for problem 1(c)

$f'(0) = -\frac{1}{2}$ $\quad y = -\frac{1}{2}x + b$, $3 = -\frac{1}{2}(0) + b$

$y = -\frac{1}{2}x + 3$ is l.

Intesect $-\frac{1}{2}x + 3 = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos$

$x = 3.38987$ $\quad y = 1.30507$

$$\int_{0}^{3.38987} \left| \left(-\frac{1}{2}x + 3\right) - \left(\frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x\right) \right| dx$$

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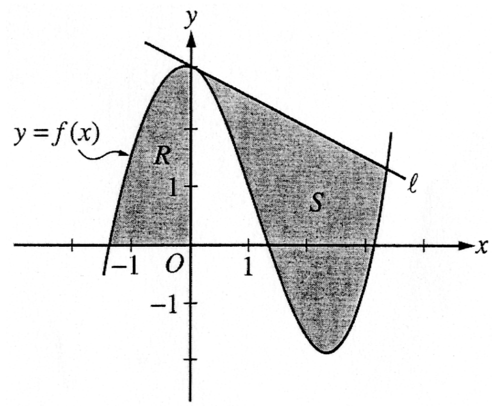
1c

CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

bounds extend from

-1.37312 to 0

$$\therefore A_R = \int_{-1.37312}^{0} \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x \, dx$$

$$\boxed{= 2.903}$$

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Continue problem 1 on page 5.

1C

Work for problem 1(b)

$$\pi\int_{-1.373/2}^{0}\left(\frac{x^3}{4}-\frac{x^2}{3}-\frac{x}{2}+3\cos x\right)^2-2^2$$

$$\boxed{=1.79\pi}$$

Work for problem 1(c)

$$A_s=\int_0^{3.1} 1-f(x)\,dx$$

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Question 1

Overview

This problem presented students with two regions. Region R was bounded in the second quadrant by a graph and the two axes. Region S was bounded by the graph and the line tangent to the graph at one point. Students needed to use integration to find two areas and a volume. In order to answer parts (a) and (b), students also had to find a zero of the function to obtain bounding values for region R. Part (a) asked students to find the area of R. Part (b) asked students to find the volume of the solid generated by rotating R about a horizontal line. In part (c) students had to find the equation of the tangent line and the x-coordinate of a point of intersection of the line and the graph in order to write an integral expression for the area of S.

Sample: 1A
Score: 9

The student earned all 9 points.

Sample: 1B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). The work in part (a) is correct. In part (b) the student earned the limits and constant point. The student writes an integral for rotation about the x-axis and does not consider the horizontal line $y = -2.$ Because of this error in the integrand, the student was not eligible for the answer point. The work in part (c) earned all 3 points.

Sample: 1C
Score: 4

The student earned 4 points: 2 points in part (a) and 2 points in part (b). The work in part (a) is correct. In part (b) the student earned the limits and constant point. The student earned 1 of the 2 integrand points. The first term of the integrand is incorrect since 2 was not added to $f(x).$ Because of this error in the integrand, the student was not eligible for the answer point. In part (c) the equation of the tangent line is not found, so the tangent line point was not earned. Since an equation for a tangent line was not found, the student could not earn the integrand point. In addition, the student did not earn the limits point for estimating the intersection point from the given graph.

Question 2

An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \tan\left(e^{-t}\right) \text{ and } \frac{dy}{dt} = \sec\left(e^{-t}\right)$$

for $t \geq 0$. At time $t = 1$, the object is at position $(2, -3)$.

(a) Write an equation for the line tangent to the curve at position $(2, -3)$.

(b) Find the acceleration vector and the speed of the object at time $t = 1$.

(c) Find the total distance traveled by the object over the time interval $1 \leq t \leq 2$.

(d) Is there a time $t \geq 0$ at which the object is on the y-axis? Explain why or why not.

(a) $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec\left(e^{-t}\right)}{\tan\left(e^{-t}\right)} = \frac{1}{\sin\left(e^{-t}\right)}$$

$$\left.\frac{dy}{dx}\right|_{(2,\,-3)} = \frac{1}{\sin\left(e^{-1}\right)} = 2.780 \text{ or } 2.781$$

$$y + 3 = \frac{1}{\sin\left(e^{-1}\right)}(x - 2)$$

$2 : \begin{cases} 1 : \left.\frac{dy}{dx}\right|_{(2,\,-3)} \\ 1 : \text{equation of tangent line} \end{cases}$

(b) $x''(1) = -0.42253,\ y''(1) = -0.15196$

$a(1) = \langle -0.423, -0.152 \rangle$ or $\langle -0.422, -0.151 \rangle$.

$$\text{speed} = \sqrt{\left(\sec\left(e^{-1}\right)\right)^2 + \left(\tan\left(e^{-1}\right)\right)^2} = 1.138 \text{ or } 1.139$$

$2 : \begin{cases} 1 : \text{acceleration vector} \\ 1 : \text{speed} \end{cases}$

(c) $$\int_1^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 1.059$$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) $$x(0) = x(1) - \int_0^1 x'(t)\, dt = 2 - 0.775553 > 0$$

The particle starts to the right of the y-axis.
Since $x'(t) > 0$ for all $t \geq 0$, the object is always moving to the right and thus is never on the y-axis.

$3 : \begin{cases} 1 : x(0) \text{ expression} \\ 1 : x'(t) > 0 \\ 1 : \text{conclusion and reason} \end{cases}$

2A

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Work for problem 2(a)

$\frac{dy}{dx}$ = slope = $\frac{dy/dt}{dx/dt}$ — ①. P (which is moving) is at x = (2, −3) at t = 1.

$\therefore \left.\frac{dx}{dt}\right|_{t=1} = \tan(1/e)$ $\left.\frac{dy}{dt}\right|_{t=1} = \sec(1/e)$

$\therefore$ slope = $\frac{\sec(1/e)}{\tan(1/e)}$ (from ①) $= \frac{1}{\sin(1/e)} = 2.780$

$\therefore$ Equation of tangent is $\boxed{y+3 = 2.780\,(x-2)}$

Work for problem 2(b)

$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}(\tan e^{-t}) = [\sec^2(e^{-t})]\,-e^{-t}$

Similarly, $\frac{d^2y}{dt^2} = \sec(e^{-t})\,\tan(e^{-t}).\,-e^{-t}$

at t = 1, acceleration vector $= \left[-\sec^2(1/e)\,e^{-1},\ -e^{-1}\sec(1/e)\tan(1/e)\right]$

$= \boxed{(-0.422,\ -0.151)}$

$\left.\text{speed}\right|_{t=1} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(\tan^2(1/e) + \sec^2(1/e))}$

$= \boxed{1.138}$

Continue problem 2 on page 7.

2A

Work for problem 2(c)

distance travelled $= \int_{t_1}^{t_2} \text{speed} \, . dt$

D from $t = 1 \to 2$ $= \int_1^2 \sqrt{[(\tan^2(e^{-t}) + (\sec^2(e^{-t})]} \, . dt$

$= \boxed{1.059}$

Work for problem 2(d)

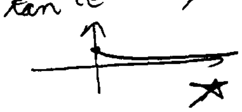
If the object (P) is on y-axis at some time, then $x = 0$ at that time, say $t = t^*$

$\therefore \quad x - 2 = \int_1^t \tan(e^{-t}) \, . dt$ $\quad \left[\Delta x = \int_1^2 dx\right]$

If $x = 0$

$\Rightarrow \quad -2 = \int_1^{t^*} \tan(e^{-t}) \, . dt$

[The curve of $\tan(e^{-t})$, $t > 0$ is



]

$\therefore$, However, since $e^{-t} > 0$ for $t > 0$ and tends to 0 as $t \to \infty$, we've that $\tan(e^{-t}) \overset{\star}{>} 0 \;\; \forall t \geq 0$

$\therefore$ The area under the curve $\tan(e^{-t})$ for $t > 0$ from $t = 1$ ~~to~~ $t = t^*$ is always > 0.

Hence, there is no $t^* > 0$ for which P is on the y-axis.

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Work for problem 2(a)

at time $t=1$, $\frac{dx}{dt} = \tan(e^{-1})$, $\frac{dy}{dt} = \sec(e^{-1})$

$$\therefore \frac{dy}{dx} = \frac{\sec(e^{-1})}{\tan(e^{-1})} = \frac{1}{\sin(e^{-1})} = 2.78$$

$\therefore$ eq. of line: $y+3 = 2.78(x-2)$

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Work for problem 2(b)

~~accell~~ $\text{acceleration} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right)$

$$= \left(\frac{-e^{-t}}{(\cos(e^{-t}))^2}, \frac{-e^{-t}\cdot\sin(e^{-t})}{(\cos(e^{-t}))^2}\right)$$

$$= (-0.42, -0.15)$$

$$\text{speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\Bigg|_{t=1} = \frac{\sqrt{(\sin(e^{-1}))^2+1}}{\cos(e^{-1})} = 1.14$$

Continue problem 2 on page 7.

Work for problem 2(c)

distance $= \int_1^2 \left(\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \right) dt$

$= 1.06$

Work for problem 2(d)

when the object is on the y-axis,
the x-coordinate of the object is 0.

the initial x-coordinate is 2,

$\frac{dx}{dt} > 0$ for all $t \geq 0$.

$\therefore$ (x-coordinate) > 0 for all $t \geq 0$.

$\therefore$ the object is not on the y-axis
for all $t \geq 0$.

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Work for problem 2(a)

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \frac{\sec(e^{-t})}{\tan(e^{-t})}$$

@ $t=1$, $\frac{dy}{dx} = 2.78058$

$y = 2.78058x + b$

$-3 = 2.78058(2) + b$

$b = -8.56115$

$\boxed{y = 2.781x - 8.561}$

Work for problem 2(b)

Speed $= \frac{dy}{dx}$ @ $t=1$ $= \frac{\sec(e^{-1})}{\tan(e^{-1})} = \boxed{2.781}$

$$\frac{d^2x}{dt^2} = \frac{-e^{-t}}{(\cos(e^{-t}))^2} \quad @\ t=1 \quad = -0.4225$$

$$\frac{d^2y}{dt^2} = \frac{-e^{-t}\sin e^{-t}}{(\cos(e^{-t}))^2} \quad @\ t=1 \quad = -0.1519$$

$\boxed{\text{acceleration } \vec{v} = (-0.423, -0.152)}$

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Continue problem 2 on page 7.

2c

Work for problem 2(c)

$$\int_1^2 \frac{\sec(e^{-t})}{\tan(e^{-t})} = \boxed{4.710}$$

Work for problem 2(d)

Object on y-axis means $x=0$

$$x(t) = \int \frac{dx}{dt} = \int \tan(e^{-t}) =$$

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Question 2

Overview

This problem dealt with particle motion in the plane. Students were given the rate of change of the x- and y-coordinates as functions of time and the initial position of the particle at time $t = 1.$ Part (a) asked for the equation of the line tangent to the curve at the point corresponding to time $t = 1.$ Part (b) asked for the acceleration vector and speed of the object at time $t = 1.$ Part (c) tested if students could use a definite integral to compute the total distance traveled by the object over a specified time interval. For part (d) students needed to make use of the initial position and sign of the first derivative for the x-coordinate to deduce that the object could never be on the y-axis. This could be done by arguing about the increasing behavior of $x(t)$ or by using the Fundamental Theorem of Calculus to express $x(t)$ in terms of a definite integral that always took positive values.

Sample: 2A
Score: 8

The student earned 8 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d). In part (b) it was not necessary for the student to compute the exact second derivatives since the values at $t = 1$ could be computed numerically on the calculator. In part (d) the student's argument is only valid for $t^* > 1.$ If $t^* < 1,$ then the student's definite integral will be negative and might equal $-2.$ The student could have completed the reasoning and earned the last point by observing that when $t^* = 0,$ the value of the definite integral is only $-0.77555.$

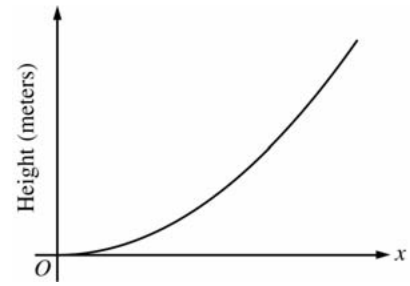
Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), 2 points in part (c), and 1 point in part (d). The work in part (a) earned both points. The acceleration vector in part (b) is computed correctly, but the answers are only reported to two decimal places rather than the three decimal places specified in the exam instructions. The student therefore did not earn the first point. The speed is also computed correctly. Since a point already had been lost for a decimal presentation error, the student was not penalized again and earned the second point in part (b). For a similar reason, the student also earned both points in part (c). In part (d) the student earned the second point for observing that $\frac{dx}{dt} > 0$ for all $t \geq 0.$ However, the student never considers the value of $x(0)$ and the reasoning is not complete, so the other 2 points were not earned.

Sample: 2C
Score: 3

The student earned 3 points: 2 points in part (a) and 1 point in part (b). The work in part (a) earned both points. The student avoids the possibility of premature rounding by using intermediate calculations to five decimal places and only rounding at the final step. In part (b) the acceleration vector is correct, but the student believes the speed is found from the slope at $t = 1.$ In part (c) the student continues to use the expression for the slope and does not earn any points. In part (d) the student sets up an integral to try to compute the position $x(t)$ but is unable to complete this approach and earned no points.

Question 3



The figure above is the graph of a function of x, which models the height of a skateboard ramp. The function meets the following requirements.

(i) At $x = 0$, the value of the function is 0, and the slope of the graph of the function is 0.
(ii) At $x = 4$, the value of the function is 1, and the slope of the graph of the function is 1.
(iii) Between $x = 0$ and $x = 4$, the function is increasing.

(a) Let $f(x) = ax^2$, where a is a nonzero constant. Show that it is not possible to find a value for a so that f meets requirement (ii) above.

(b) Let $g(x) = cx^3 - \frac{x^2}{16}$, where c is a nonzero constant. Find the value of c so that g meets requirement (ii) above. Show the work that leads to your answer.

(c) Using the function g and your value of c from part (b), show that g does not meet requirement (iii) above.

(d) Let $h(x) = \frac{x^n}{k}$, where k is a nonzero constant and n is a positive integer. Find the values of k and n so that h meets requirement (ii) above. Show that h also meets requirements (i) and (iii) above.

(a) $f(4) = 1$ implies that $a = \frac{1}{16}$ and $f'(4) = 2a(4) = 1$ implies that $a = \frac{1}{8}$. Thus, f cannot satisfy (ii).	$2: \begin{cases} 1: a = \frac{1}{16} \text{ or } a = \frac{1}{8} \\ 1: \text{shows } a \text{ does not work} \end{cases}$
(b) $g(4) = 64c - 1 = 1$ implies that $c = \frac{1}{32}$. When $c = \frac{1}{32}$, $g'(4) = 3c(4)^2 - \frac{2(4)}{16} = 3\left(\frac{1}{32}\right)(16) - \frac{1}{2} = 1$	1 : value of c
(c) $g'(x) = \frac{3}{32}x^2 - \frac{x}{8} = \frac{1}{32}x(3x - 4)$; $g'(x) < 0$ for $0 < x < \frac{4}{3}$, so g does not satisfy (iii).	$2: \begin{cases} 1: g'(x) \\ 1: \text{explanation} \end{cases}$
(d) $h(4) = \frac{4^n}{k} = 1$ implies that $4^n = k$. $h'(4) = \frac{n4^{n-1}}{k} = \frac{n4^{n-1}}{4^n} = \frac{n}{4} = 1$ gives $n = 4$ and $k = 4^4 = 256$. $h(x) = \frac{x^4}{256} \Rightarrow h(0) = 0$. $h'(x) = \frac{4x^3}{256} \Rightarrow h'(0) = 0$ and $h'(x) > 0$ for $0 < x < 4$.	$4: \begin{cases} 1: \frac{4^n}{k} = 1 \\ 1: \frac{n4^{n-1}}{k} = 1 \\ 1: \text{values for } k \text{ and } n \\ 1: \text{verifications} \end{cases}$

Work for problem 3(a)

according to (ii), $f(4) = 1$, $f'(4) = 1$

$f(x) = ax^2 \rightarrow 16a = 1 \quad a = \frac{1}{16}$

$f'(x) = 2ax \rightarrow 8x = 1 \quad a = \frac{1}{8}$

$\frac{1}{16} \neq \frac{1}{8}$

$\therefore$ it's impossible to find a value for a so that f meets requirement (ii).

Work for problem 3(b)

according to (ii), $g(4) = 1$, $g'(4) = 1$

$g(x) = cx^3 - \frac{x^2}{16} \rightarrow 64c - \frac{16}{16} = 64c - 1 = 1 \qquad c = \frac{1}{32}$

$g'(x) = 3cx^2 - \frac{1}{8}x \rightarrow 3 \cdot 16 \cdot c - \frac{1}{2} = 48c - \frac{1}{2} = 1 \qquad c = \frac{1}{32}$

$\therefore c = \frac{1}{32}$

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Continue problem 3 on page 9

Work for problem 3(c)

$g'(x) = \frac{3}{32}x^2 - \frac{1}{8}x = \frac{3}{32}x(x - \frac{4}{3})$

$\therefore$ $x < 0$: $g'(x) > 0$, $g(x)$ increasing

$0 < x < \frac{4}{3}$: $g'(x) < 0$, $g(x)$ decreasing

$\frac{4}{3} < x$: $g'(x) > 0$, $g(x)$ increasing

$\therefore$ $g(x)$ do not increase when $0 < x < \frac{4}{3}$. So it does not meet requirement (iii)

Work for problem 3(d)

according to (ii), $h(4) = 1$, $h'(4) = 1$

$h(x) = \frac{x^n}{k} \rightarrow \frac{4^n}{k} = 1$

$h'(x) = \frac{n}{k} \cdot x^{n-1} \rightarrow \frac{n}{k} \cdot 4^{n-1} = 1$

$\therefore$ $4^n = k$, $4^{n-1} \cdot n = k$

$\therefore n = 4 \quad k = 256$

$\therefore h(x) = \frac{x^4}{256}$

$h(0) = 0$, $h'(0) = 0$ $\rightarrow$ meet requirement (i)

$h'(x) = \frac{4}{256} \cdot x^3 = \frac{1}{64}x^3 \quad x > 0$, $h'(x) > 0$ $\therefore h(x)$ increasing $\rightarrow$ meet requirement (iii).

END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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Work for problem 3(a)

$f(x) = ax^2$

$f'(x) = 2ax$

to satisfy (ii), $x = 4$

$f(4) = 1$

$f'(4) = 1$

$f(x) = 16a = 1$

$a = \frac{1}{16}$

$f'(x) = 2 \cdot 4a = 1$

$a = \frac{1}{8}$

- There is no value a that satisfies requirement (ii)

Work for problem 3(b)

$g(x) = cx^3 - \frac{x^2}{16}$ $\quad g(4) = 64c - 1 = 1 \quad c = \frac{1}{32}$

$g'(x) = 3cx^2 - \frac{x}{8}$ $\quad g'(4) = 48c - .5 = 1 \quad c = \frac{1}{32}$

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Continue problem 3 on page 9.

Work for problem 3(c)

$g(x) = \frac{1}{32}x^3 - \frac{x^2}{16}$ $\quad$ $g'(0) = 0$

$g'(x) = \frac{3}{32}x^2 - \frac{x}{8} = 0$ $\quad$ $g'(4) = 1$

$x\left(\frac{3}{32}x - \frac{1}{8}\right) = 0$

$x = 0 \quad x = 1.333$

Because $g'(0) = 0$, $g(x)$ is not increasing at $x=0$, thus it does not satisfy requirement → (iii)

Work for problem 3(d)

$h(x) = \frac{x^n}{k}$ $\quad$ $\frac{4^n}{k} = 1 \quad 4^n = k$

$h'(x) = \frac{nx^{n-1}}{k}$ $\quad$ $\frac{n4^{n-1}}{k} = 1 \quad n4^{n-1} = k$

$4^n = n4^{n-1}$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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Work for problem 3(a)

$f(x) = ax^2$

$y = ax^2$

$1 = 16a$

$a = \frac{1}{16}$

$y = \frac{1}{16}x^2$

$a = \frac{1}{16}$

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Work for problem 3(b)

$x = 4$
$y = 1$

$1 = 64c - 1$

$2 = 64c$

$c = \frac{1}{32}$

Continue problem 3 on page 9

3C

Work for problem 3(c)

$$g(x) = \frac{x^3}{32} - \frac{x^2}{16}$$

$$= \frac{x^3 - 2x^2}{32}$$

$x = 0$
$y = 0$

$x = 1$
$y = -\frac{1}{32}$

$x = 2$
$y = 0$

$x = 3$
$y = 0$

$x = 4$
$y = 1$

Work for problem 3(d)

$$h(x) = \frac{x^n}{k}$$

$$1 = \frac{4^n}{k}$$

$$k = 4^n.$$

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END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Question 3

Overview

This problem presented three requirements that had to be satisfied by the graph of a function modeling the height of a skateboard ramp. Students were asked to investigate three families of functions that might be used for such a model. In part (a) they were asked to show that no quadratic of the form ax^2 would satisfy the second requirement. In part (b) they were asked to find the coefficient c for which the cubic $cx^3 - \frac{x^2}{16}$ would meet the second requirement, but then show in part (c) that the cubic with this value of c does not meet the third requirement. Finally, in part (d) students were asked to find the values of n and k for which the power function $\frac{x^n}{k}$ would meet all three requirements.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

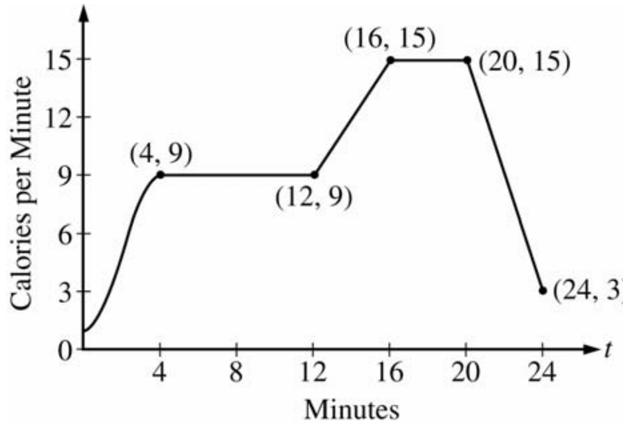
The student earned 6 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and 2 points in part (d). The student's work is correct in parts (a) and (b). In part (c) the student earned 1 point for finding the derivative of g. The student does not explain why g is not increasing between $x = 0$ and $x = 4$ and so did not earn the second point in this part. In part (d) the student sets up correct equations to find n and k, earning 1 point for each equation, but does not find n or k and thus cannot show that the function h meets requirements (i) and (iii).

Sample: 3C
Score: 3

The student earned 3 points: 1 point in part (a), 1 point in part (b), and 1 point in part (d). In part (a) the student finds the value of a for which $f(4) = 1$, which earned the first point, but fails to show that this value of a does not work to meet requirement (ii). In part (b) the student uses the information about g to find the desired value of c. In part (c) the student's calculations of the values of the function g at integer values of x earned no points (and the value at $x = 3$ is incorrect). However, both points could have been earned in part (c) with those calculations if the student had gone on to observe that the value of y at $x = 1$ is less than the value of y at $x = 0$, and hence the function g is not increasing on the interval $0 \le x \le 4$. In part (d) the student earned 1 point for using the information about $h(4)$ to write an equation for n and k but has no other work.

Question 4

The rate, in calories per minute, at which a person using an exercise machine burns calories is modeled by the function f. In the figure above, $f(t) = -\frac{1}{4}t^3 + \frac{3}{2}t^2 + 1$ for $0 \le t \le 4$ and f is piecewise linear for $4 \le t \le 24$.



(a) Find $f'(22)$. Indicate units of measure.

(b) For the time interval $0 \le t \le 24$, at what time t is f increasing at its greatest rate? Show the reasoning that supports your answer.

(c) Find the total number of calories burned over the time interval $6 \le t \le 18$ minutes.

(d) The setting on the machine is now changed so that the person burns $f(t) + c$ calories per minute. For this setting, find c so that an average of 15 calories per minute is burned during the time interval $6 \le t \le 18$.

(a) $f'(22) = \dfrac{15-3}{20-24} = -3$ calories/min/min

1 : $f'(22)$ and units

(b) f is increasing on $[0, 4]$ and on $[12, 16]$.

On $(12, 16)$, $f'(t) = \dfrac{15-9}{16-12} = \dfrac{3}{2}$ since f has constant slope on this interval.

On $(0, 4)$, $f'(t) = -\dfrac{3}{4}t^2 + 3t$ and

$f''(t) = -\dfrac{3}{2}t + 3 = 0$ when $t = 2$. This is where f' has a maximum on $[0, 4]$ since $f'' > 0$ on $(0, 2)$ and $f'' < 0$ on $(2, 4)$.

On $[0, 24]$, f is increasing at its greatest rate when $t = 2$ because $f'(2) = 3 > \dfrac{3}{2}$.

$$4 : \begin{cases} 1 : f' \text{ on } (0, 4) \\ 1 : \text{shows } f' \text{ has a max at } t = 2 \text{ on } (0, 4) \\ 1 : \text{shows for } 12 < t < 16,\ f'(t) < f'(2) \\ 1 : \text{answer} \end{cases}$$

(c) $\displaystyle\int_6^{18} f(t)\,dt = 6(9) + \frac{1}{2}(4)(9+15) + 2(15)$
$= 132$ calories

$$2 : \begin{cases} 1 : \text{method} \\ 1 : \text{answer} \end{cases}$$

(d) We want $\displaystyle\frac{1}{12}\int_6^{18} (f(t) + c)\,dt = 15.$

This means $132 + 12c = 15(12)$. So, $c = 4$.

OR

Currently, the average is $\dfrac{132}{12} = 11$ calories/min.

Adding c to $f(t)$ will shift the average by c.
So $c = 4$ to get an average of 15 calories/min.

$$2 : \begin{cases} 1 : \text{setup} \\ 1 : \text{value of } c \end{cases}$$

NO CALCULATOR ALLOWED

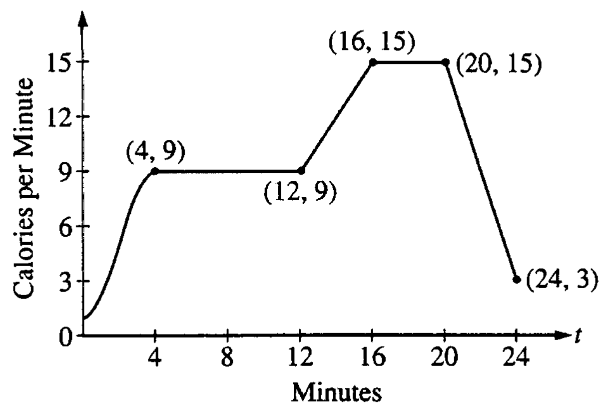
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



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Work for problem 4(a)

$f'(22) =$ Gradient of straight line from $t=20$ to $t=24$

$= \frac{3-15}{24-20}$

$= \frac{-12}{4}$

$= -3$ calories/minute^2

Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

Work for problem 4(b)

From graph, we see that f is only increasing in the intervals $0 \leq t \leq 4$ and $12 \leq t \leq 16$

Rate of increment for $12 \leq t \leq 16$

$= \frac{15-9}{16-12}$

$= \frac{6}{4} = 1.5$

For $0 \leq t \leq 4$,

$f'(t) = -\frac{3}{4}t^2 + 3t$

$f''(t) = -\frac{3}{2}t + 3$

At greatest rate of increment, $f''(t) = 0 \Rightarrow t = 2$

$f'''(t) = -\frac{3}{2} < 0 \Rightarrow$ At $t=2$, rate of increment is greatest and not smallest

$f'(2) = -\frac{3}{4}(2)^2 + 3(2) = 3 > 1.5$

$\therefore f$ is increasing at its greatest rate ~~than~~ at $t = 2$

Work for problem 4(c)

~~For 6 ≤ t ≤ 12~~

~~For 6 ≤ t ≤ 12~~,

Total number of calories burned $= \int_6^{18} f(t)\,dt$

~~$= \int_6^{12} 9\,dt + \int_{12}^{16}$~~

$= 9(12-6) + \frac{1}{2}(9+15)(16-12) + 15(18-16)$

$= 54 + 48 + 30 = 132$ calories

Work for problem 4(d)

Before setting is changed, average calories in $6 \leq t \leq 18$

$= \frac{1}{18-6}\int_6^{18} f(t)\,dt = \frac{132}{12} = 11$ calories

Now, $\frac{1}{18-6}\int_6^{18}[f(t)+c]\,dt = 15$

$\Rightarrow \frac{1}{12}\int_6^{18} f(t)\,dt + \frac{1}{12}\int_6^{18} c\,dt = 15$

$\Rightarrow \frac{1}{12}[cx]_6^{18} = 15 - 11 = 4$

$\frac{1}{12}(18-6)c = 4$

$\Rightarrow c = 4$

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NO CALCULATOR ALLOWED

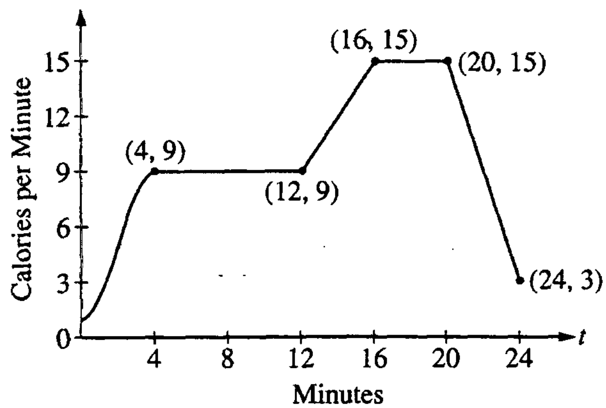
CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Work for problem 4(a)

$$f'(22) = \frac{f(24)-f(20)}{24-20} = \frac{3-15}{4} = -\frac{12}{4} = -3$$

$\therefore\ -3\,\text{cal}/\text{min}^2$

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Continue problem 4 on page 11

NO CALCULATOR ALLOWED

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Work for problem 4(b)

i) $0 \le t \le 4$, $f'(t) = -\frac{3}{4}t^2 + 3t$

$= -\frac{3}{4}(t^2 - 4t + 4) + 3$

$= -\frac{3}{4}(t-2)^2 + 3$

ii) $f'(t) = 0$ for $4 \le t < 12$

iii) $f'(t) = \frac{15-9}{16-12} = \frac{6}{4} = \frac{3}{2}$ for $12 \le t < 16$

iv) $f'(t) = 0$ for $16 \le t < 20$

v) $f'(t) < 0$ for $20 \le t < 24$

$\therefore$ $f'(t)$ is the greatest when $t=2$, which also means that $f(t)$ is increasing at its greatest rate

$\therefore t=2$.

Work for problem 4(c)

i) $6 \le t < 12$ $f(t)=9$: $6 \times 9 = 54$

ii) $12 \le t < 16$ $\frac{1}{2} \times (9+15) \times 4 = 48$

iii) $16 \le t \le 18$ $f(t)=15$ $2 \times 15 = 30$

$\therefore 54 + 48 + 30 = 132$ $\therefore$ 132 calories

Work for problem 4(d)

$132 + c \times (18-6) = 132 + 12c$

Since $\frac{132 + 12c}{12} = 15$, $c = \frac{180 - 132}{12} = 4$

$\therefore c = 4$

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NO CALCULATOR ALLOWED

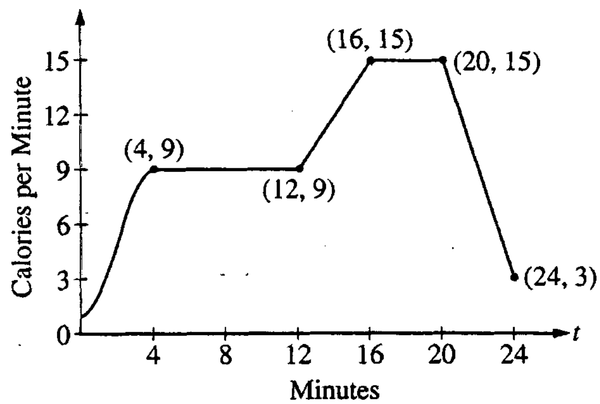
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



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Work for problem 4(a)

function for $20 \leq x \leq 24$

$m = \frac{3-15}{24-20}$ $\quad y-3=-3(x-24)$

$\frac{-12}{4}$ $\quad y=-3x+75$

$m=-3$

$f(x)=-3x+75$

$f'(22) = -3$ calories/minutes^2

Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

Work for problem 4(b)

function increases: at $0 \le x < 4$, $12 \le x \le 16$

$f(t) = -\frac{1}{4}t^3 + \frac{3}{2}t^2 + 1$

$f'(t) = -\frac{3}{4}t^2 + 3t$

$f''(t) = -\frac{3}{2}t + 3 = 0$

$t = 2$

maximum for $f'(t)$ at $t = 2$

$f'(2) = -\frac{3}{4}(4) + 6$

$= 3$

$m = \frac{15-9}{16-12} = \frac{3}{2}$

$f(x) = y - 15 = \frac{3}{2}(x-16)$

$f(x) = \frac{3}{2}x - 9$

$f'(x) = \frac{3}{2}$

Work for problem 4(c)

Calories total $= \int_6^{12} 9\,dx + \int_{12}^{16} (\frac{3}{2}x - 9)\,dx + \int_{16}^{18} 15\,dx$

$9x\Big]_6^{12} + \frac{3}{4}x^2 - 9\Big]_{12}^{16} + 15x\Big]_{16}^{18}$

$105 - 54 + 183 - 99 + 270 - 240$

$54 + 30 + 84$

164 calories burned

Work for problem 4(d)

$15 = \frac{\int_6^{18}(f(x)+c)\,dx}{12}$

$180 = \int_6^{18} f(x)\,dx + \int_6^{18} c\,dx$

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Question 4

Overview

This problem presented students with a piecewise-defined function f that modeled the rate at which a person using an exercise machine burns calories. The graph of f consisted of a cubic part and a part that was piecewise linear. In part (a) students were asked to find $f'(22)$, which required them to recognize the relationship between this value and the slope of one of the line segments in the graph of f. It was also important to use correct units. For part (b) students had to consider the two parts of the graph where f was increasing and determine the time when f was increasing at its greatest rate. In part (c) students had to use a definite integral to find the total number of calories burned over a given time interval. The evaluation of the definite integral could be done using geometry since the graph over the given time interval consisted of two line segments, one of which was horizontal. For part (d) students were expected to use the value of their integral from part (c) to work with the average value of the function f shifted up by c calories per minute.

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 1 point in part (a), 1 point in part (b), 2 points in part (c), and 2 points in part (d). The student shows correct work for parts (a), (c), and (d). In part (b) the first derivative is correct and the student earned the first point. The student neither explains why $f'(t)$ has a maximum at 2 nor states the value of $f'(2)$. A proper reasoning for the final answer is not given, and the student did not earn any more points.

Sample: 4C
Score: 4

The student earned 4 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and 1 point in part (d). The work in part (a) is correct. In part (b) the first derivative is correct, so the student earned the first point. The student neither explains why $f'(t)$ has a maximum at 2 in the interval $0 \le t \le 4$ nor shows a comparison among the values of $f'(t)$. The student does not provide a final answer and did not earn any more points. In part (c) the setup is correct. The antiderivative of the second integral is incorrect, so the student did not earn the answer point. In part (d) the setup is correct, but the student does not finish the problem and could not earn the second point.

Question 5

Let f be a function with $f(4) = 1$ such that all points (x, y) on the graph of f satisfy the differential equation

$$\frac{dy}{dx} = 2y(3 - x).$$

Let g be a function with $g(4) = 1$ such that all points (x, y) on the graph of g satisfy the logistic differential equation

$$\frac{dy}{dx} = 2y(3 - y).$$

(a) Find $y = f(x)$.

(b) Given that $g(4) = 1$, find $\lim_{x\to\infty} g(x)$ and $\lim_{x\to\infty} g'(x)$. (It is not necessary to solve for $g(x)$ or to show how you arrived at your answers.)

(c) For what value of y does the graph of g have a point of inflection? Find the slope of the graph of g at the point of inflection. (It is not necessary to solve for $g(x)$.)

(a) $\frac{dy}{dx} = 2y(3 - x)$

$\frac{1}{y}\,dy = 2(3 - x)\,dx$

$\ln|y| = 6x - x^2 + C$

$0 = 24 - 16 + C$

$C = -8$

$\ln|y| = 6x - x^2 - 8$

$y = e^{6x - x^2 - 8}$ for $-\infty < x < \infty$

$$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solution} \end{cases}$$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

(b) $\lim_{x\to\infty} g(x) = 3$

$\lim_{x\to\infty} g'(x) = 0$

$$2 : \begin{cases} 1 : \lim_{x\to\infty} g(x) = 3 \\ 1 : \lim_{x\to\infty} g'(x) = 0 \end{cases}$$

(c) $\frac{d^2y}{dx^2} = (6 - 4y)\frac{dy}{dx}$

Because $\frac{dy}{dx} \neq 0$ at any point on the graph of g, the concavity only changes sign at $y = \frac{3}{2}$, half the carrying capacity.

$\left.\frac{dy}{dx}\right|_{y=3/2} = 2\left(\frac{3}{2}\right)\left(3 - \frac{3}{2}\right) = \frac{9}{2}$

$$2 : \begin{cases} 1 : y = \frac{3}{2} \\ 1 : \left.\frac{dy}{dx}\right|_{y=3/2} \end{cases}$$

5A

NO CALCULATOR ALLOWED

Work for problem 5(a)

$\frac{dy}{dx}=2y(3-x)$

$dy=2y(3-x)dx$

$\int\frac{1}{2y}dy=\int(3-x)dx$

$\frac{1}{2}\ln|y|=3x-\frac{x^2}{2}+C$

$\ln|y|=2\left(3x-\frac{x^2}{2}+C\right)$

$\ln|y|=6x-x^2+C$

$y=e^{6x-x^2+C}$

$y=Ce^{6x-x^2}$

$f(4)=1=Ce^{24-16}=Ce^8$

$\frac{1}{e^8}=\frac{C\cancel{e^8}}{\cancel{e^8}}$

$\boxed{y=f(x)=\frac{1}{e^8}e^{6x-x^2}}$

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Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

Work for problem 5(b)

$\frac{dP}{dt} = ky(L-y)$

L represents carrying capacity

$\frac{dy}{dx} = 2y(3-y)$

3 is the carrying capacity

$\lim_{x\to\infty} g(x) = $ 3

$\lim_{x\to\infty} g'(x) = $ 0

Work for problem 5(c)

$\lim_{x\to\infty} g(x) = 3$

$\frac{dy}{dx} = 2y(3-y)$

This is a logistic function with carrying capacity of 3

There is always a point of inflection in logistic functions whenever y is half of the carrying capacity

∴ $\boxed{y = 1.5 \text{ is the inflection point}}$

$\left.\frac{dy}{dx}\right|_{y=1.5} = 2(1.5)(3-1.5)$

$= 3(1.5) = \boxed{4.5}$

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Work for problem 5(a)

$\frac{dy}{dx} = 2y(3-x)$

$\frac{dy}{2y} = (3-x)\,dx$

$\int \frac{dy}{2y} = \int (3-x)\,dx$

$\frac{1}{2}\ln|y| = 3x - \frac{x^2}{2} + c$

$\ln|y| = 6x - x^2 + c$

$y = e^{6x-x^2+c}$ $(e^c = C)$

$= Ce^{6x-x^2}$

$f(4) = 1$

$1 = Ce^8$

$\therefore C = \frac{1}{e^8} = e^{-8}$

$\therefore y = e^{-8} \cdot e^{6x-x^2}$

$= e^{6x-x^2-8}$

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Continue problem 5 on page 13.

5B

NO CALCULATOR ALLOWED

Work for problem 5(b)

$$\frac{dy}{2y(3-y)} = dx$$

$$\frac{dy}{6y} + \int \frac{dy}{3(6-2y)} = \int dx$$

$$\frac{1}{6}\ln|y| + -\frac{1}{6}\ln|6-2y| = x + C$$

$$\ln y - \ln|6-2y| = 6x + C$$

$$\ln\left|\frac{y}{6-2y}\right| = 6x + C$$

$$\ln\left|\frac{6-2y}{y}\right| = -6x + C$$

$$\frac{6-2y}{y} = e^{-6x+c}$$

$$= Ce^{-6x} \quad (C = \pm e^{c})$$

$\lim_{x\to\infty} g(x) =$ 4

$\lim_{x\to\infty} g'(x) =$ 0

$\boxed{g(4)=1} \rightarrow 4 = Ce^{-24}$ ~~$g(x) = 4e^{24}e^{-24}$~~ $= 4$

Work for problem 5(c)

point of inflection $= g''(x) = 0$. ~~and change~~ (change its sign)

$$\left(\frac{dy}{dx}\right)' = (6y - 2y^2)'$$

$$= 6\frac{dy}{dx} - 4y\frac{dy}{dx}$$

$$= 6(6y-2y^2) - 4y(6y-2y^2)$$

$$= 36y - 12y^2 - 24y^2 + 8y^3$$

$$= 8y^3 - 36y^2 + 36y = 0$$

$y = 0, \frac{3}{2}, 3$

5C

NO CALCULATOR ALLOWED

Work for problem 5(a)

$$\frac{dy}{dx} = 2y(3-x)$$

$$\int dy = \int [2y(3-x)]\,dx$$

$$y = \int [6y - 2xy]\,dx$$

$$y = f(x) = 6xy - x^2y + C$$

$$f(4) = 6 \times 4 \times 1 - 4^2 \times 1 + C$$

$$= 24 - 16 + C = 1$$

$$C = -8$$

$$\therefore \; f(x) = 6xy - x^2y - 8$$

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Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

Work for problem 5(b)

$\lim_{x \to \infty} g(x) = \frac{1}{3}$

$\lim_{x \to \infty} g'(x) = 0$

Work for problem 5(c)

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In the point of inflection, $\frac{d^2y}{dx^2} = 0$ and the concavity of $g(x)$ changes.

(i) $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(6y - 2y^2)$

$= 6 \times \frac{dy}{dx} - 4y \times \frac{dy}{dx}$

$= (6-4y) \times \frac{dy}{dx}$

$= (6-4y)(2y(3-y)) = 0$

$\therefore y = \frac{3}{2}$ ($y \neq 3$ and $y \neq 0$, since the concavity of $g(x)$ does not change when $\frac{dy}{dx} = 0$)

(ii) slope of graph g at $y = 3/2$.

$\therefore \frac{dy}{dx} = 2y(3-y) = 2 \times 3/2\left(3 - \frac{3}{2}\right) = 3 \times \frac{3}{2} = \frac{9}{2}$

Answer: $y = \frac{3}{2}$; slope of graph g at the point of inflection is $\frac{9}{2}$

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Question 5

Overview

This problem asked students to work with two differential equations. In part (a) students had to find the particular solution $f(x)$ to a separable differential equation satisfying a given initial condition. Parts (b) and (c) tested students' knowledge of the behavior of a solution $g(x)$ to a logistic differential equation that was superficially similar to, but in fact quite different from, the differential equation solved in part (a). It was not necessary for students to solve for this particular solution. Part (b) tested their knowledge of the limiting behavior of a logistic equation: if the initial y value is positive, then $g(x)$ will approach the carrying capacity, which is the positive root of the quadratic polynomial in y, and $g'(x)$ will approach 0. Even without specific knowledge of the actual solution, students should have been able to determine this information directly from the logistic differential equation. Part (c) tested student recognition that for this particular solution, the point of inflection for the graph of g will occur where $g'(x)$ is greatest. This occurs at the maximum value of the quadratic polynomial in y, or halfway between the two roots (half the carrying capacity). The graph of this particular solution will have a point of inflection since the initial y value is less than half the carrying capacity. The slope at this point could be computed directly from the differential equation.

Sample: 5A
Score: 9

The student earned all 9 points. In parts (b) and (c) the student uses knowledge of the general properties of the solution to a logistic differential equation to help in answering questions about the limits and the point of inflection.

Sample: 5B
Score: 6

The student earned 6 points: all 5 points in part (a) and 1 point in part (b). In part (b) the student attempts to solve the logistic differential equation using the technique of partial fractions. This is not necessary, and that work was not graded. The student does write the correct limit of the derivative and earned 1 point. In part (c) the student uses implicit differentiation and the chain rule to compute the second derivative. The student reports all three zeros as the answer, however, and thus did not earn the first point. The student could still have earned both points in part (c) by identifying $y = \frac{3}{2}$ as the only value of y where the graph of g has an inflection point and computing the slope at that value.

Sample: 5C
Score: 3

The student earned 3 points: 1 point in part (b) and 2 points in part (c). In part (a) the student does not separate the variables. The student attempts to antidifferentiate an expression that still contains both the y and x variables and therefore received none of the 5 available points. In part (b) the student has the correct limit for the derivative, but the answer for the limit of the function is the reciprocal of the carrying capacity. The work in part (c) earned both points.

Question 6

The function f is defined by $f(x) = \dfrac{1}{1+x^3}$. The Maclaurin series for f is given by

$$1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots,$$

which converges to $f(x)$ for $-1 < x < 1$.

(a) Find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

(b) Use your results from part (a) to find the sum of the infinite series $-\dfrac{3}{2^2} + \dfrac{6}{2^5} - \dfrac{9}{2^8} + \cdots + (-1)^n \dfrac{3n}{2^{3n-1}} + \cdots$.

(c) Find the first four nonzero terms and the general term for the Maclaurin series representing $\int_0^x f(t)\,dt$.

(d) Use the first three nonzero terms of the infinite series found in part (c) to approximate $\int_0^{1/2} f(t)\,dt$. What are the properties of the terms of the series representing $\int_0^{1/2} f(t)\,dt$ that guarantee that this approximation is within $\dfrac{1}{10{,}000}$ of the exact value of the integral?

(a) $f'(x) = -3x^2 + 6x^5 - 9x^8 + \cdots + 3n(-1)^n x^{3n-1} + \cdots$	2 : { 1 : first three terms; 1 : general term }
(b) The given series is the Maclaurin series for $f'(x)$ with $x = \frac{1}{2}$. $f'(x) = -\left(1+x^3\right)^{-2}\left(3x^2\right)$. Thus, the sum of the series is $f'\left(\frac{1}{2}\right) = -\dfrac{3\left(\frac{1}{4}\right)}{\left(1+\frac{1}{8}\right)^2} = -\dfrac{16}{27}$.	2 : { 1 : $f'(x)$; 1 : $f'\left(\frac{1}{2}\right)$ }
(c) $\displaystyle\int_0^x \frac{1}{1+t^3}\,dt = x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots + \frac{(-1)^n x^{3n+1}}{3n+1} + \cdots$	2 : { 1 : first four terms; 1 : general term }
(d) $\displaystyle\int_0^{1/2} \frac{1}{1+t^3}\,dt \approx \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7}$. The series in part (c) with $x = \frac{1}{2}$ has terms that alternate, decrease in absolute value, and have limit 0. Hence the error is bounded by the absolute value of the next term. $\displaystyle\left\lvert \int_0^{1/2} \frac{1}{1+t^3}\,dt - \left(\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7}\right)\right\rvert < \frac{\left(\frac{1}{2}\right)^{10}}{10} = \frac{1}{10240} < 0.0001$	3 : { 1 : approximation; 1 : properties of terms; 1 : absolute value of fourth term < 0.0001 }

NO CALCULATOR ALLOWED

Work for problem 6(a)

$f'(x) = -3x^2 + 6x^5 - 9x^8 + \ldots + (-1)^n(3n)x^{3n-1} + \ldots$

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Work for problem 6(b)

$\therefore \boxed{x = \frac{1}{2}}$ for this series, if we let $x = \frac{1}{2}$, we get:

$\frac{-3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \ldots + \frac{(-1)^n(3n)}{2^{3n-1}} + \ldots$

this is the derivative of $f'(x)$ at $x = \frac{1}{2}$

$f'(x) = \frac{-3x^2}{(1+x^3)^2}$

$f'\left(\frac{1}{2}\right) = \frac{-3\left(\frac{1}{2}\right)^2}{\left(1+\left(\frac{1}{2}\right)^3\right)^2}$ $= \frac{-3\left(\frac{1}{4}\right)}{\left(1+\frac{1}{8}\right)^2} = \frac{-\frac{3}{4}}{\left(\frac{9}{8}\right)^2} = \frac{-\frac{3}{4}}{\frac{81}{64}} = \boxed{\frac{-3(64)}{4(81)}}$ sum of the series

Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

Work for problem 6(c)

$$\int_0^x f(t)\,dt = x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{1}{10}x^{10} + \ldots + \frac{(-1)^n x^{3n+1}}{(3n+1)} + \ldots - (0)$$

$$\int_0^x f(t)\,dt = x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{1}{10}x^{10} + \ldots + \frac{(-1)^n x^{3n+1}}{(3n+1)} + \ldots$$

Work for problem 6(d)

$$\int_0^{1/2} f(t)\,dt = \left[x - \frac{1}{4}x^4 + \frac{1}{7}x^7\right]_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{4}\cdot\frac{1}{16} + \frac{1}{7}\cdot\frac{1}{128} - (0)$$

$$= \boxed{\frac{1}{2} - \frac{1}{64} + \frac{1}{896}}$$

this is an alternating series, so the error is less than or equal to the next unused term

$$\text{error} \le \frac{1}{10}\left(\frac{1}{2^{10}}\right)$$

$$= \frac{1}{10}\cdot\frac{1}{1024}$$

$$= \boxed{\frac{1}{10240}}$$

this is the error, and it is less than $\frac{1}{10000}$, thus the integral value here is within $\frac{1}{10000}$ of the exact integral

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Work for problem 6(a)

$f'(x) = -3x^2 + 6x^5 - 9x^8$

Work for problem 6(b)

$-\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \dots + (-1)^n \frac{3n}{2^{3n-1}} = f'\left(\frac{1}{2}\right)$

$f'(x) = \frac{-3x^2}{(1+x^3)^2}$ $\therefore f'\left(\frac{1}{2}\right) = \frac{-3\left(\frac{1}{4}\right)}{\left(\frac{9}{8}\right)^2} = \frac{-3}{4} \div \frac{81}{64}$

$= \frac{-\cancel{3}^{1}}{\cancel{4}_{1}} \times \frac{\cancel{64}^{16}}{\cancel{81}_{27}} = -\frac{16}{27}$

Continue problem 6 on page 15.

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Work for problem 6(c)

$$\int_0^x f(t) = \int_0^x 1 - x^3 + x^6 - x^9 \ldots\ dt$$

$$= x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} \ldots$$

$\therefore$ first four non zero term $\int_0^x f(t)$ is $x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10}$

Work for problem 6(d)

$$\int_0^{1/2} f(t) = \frac{1}{2} - \frac{1}{4(2)^4} + \frac{1}{7(2)^7} = \frac{1}{2} - \frac{1}{64} + \frac{1}{896} = \frac{448}{896} - \frac{14}{896} + \frac{1}{896}$$

$$= \frac{435}{896}$$

because the absolute. of 4th non zero term is less than $\frac{1}{10000}$
(test the error)

4th term $= \left|\frac{1}{2^{10} \times 10}\right| = \frac{1}{10240} \Rightarrow$ less than $\frac{1}{10000}$

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Work for problem 6(a)

$f(x) = 1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n}$

$f'(x) = -3x^2 + 6x^5 - 9x^8 + \cdots + 3n(-1)^n x^{(3n-1)} + \cdots$
(first 3)

Work for problem 6(b)

When $x = \frac{1}{2}$,

(a) ~~series~~ meets (b)'s series.

$f'(\frac{1}{2}) = -\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \cdots + (-1)^n \frac{3n}{2^{3n-1}}$

$= -\frac{3}{4} + \frac{6}{32} - \frac{9}{256} + \cdots$

$\times(-\frac{1}{4})$ $\times(-\frac{3}{16})$

$\times 3$

Continue problem 6 on page 15.

 6c

NO CALCULATOR ALLOWED

Work for problem 6(c)

$\int_0^x f(t)\,dt$

$= x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^9}{9} + \cdots + (-1)^n \cdot \frac{x^{3n+1}}{3n+1} + \cdots$

Work for problem 6(d)

$\int_0^{\frac{1}{2}} f(t)\,dt$

because this series is decreasing alternating series,

$\left|\left(\int_0^{\frac{1}{2}} f(t)\,dt\right) - \left[x - \frac{x^4}{4} + \frac{x^7}{7}\right]_0^{\frac{1}{2}}\right| \leq \underline{\left|\left[-\frac{x^9}{9}\right]_0^{\frac{1}{2}}\right|}$

$\hookrightarrow \underline{\underline{\frac{1}{512 \times 9} < \frac{1}{10000}}}$

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Question 6

Overview

This problem presented students with the Maclaurin series for $f(x) = \frac{1}{1+x^3}$. In part (a) students were asked to find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$. In part (b) they then needed to use their results to find the sum of the infinite series that is obtained by evaluating the Maclaurin series for $f'(x)$ at $x = \frac{1}{2}$. Part (c) asked for the first four nonzero terms and the general term for the Maclaurin series for $\int_0^{1/2} f(t)\,dt$, and then part (d) asked students to use the first three nonzero terms to approximate this definite integral. The second question in part (d) asked students to list the properties of the terms of the series representing this definite integral that guaranteed that their approximation is within $\frac{1}{10{,}000}$ of the exact value of the integral. This question tested whether students knew that the terms must not only alternate and converge to 0, but must also decrease in absolute value to use the error bound for alternating series.

Sample: 6A
Score: 8

The student earned 8 points: 2 points in each of parts (a), (b), (c), and (d). The student correctly reports the first three nonzero terms and the general term in the Maclaurin series for $f'(x)$ in part (a). In part (b) the derivative $f'(x)$ and the special value $f'\left(\frac{1}{2}\right)$ are both computed correctly. In part (c) the student correctly reports the first four nonzero terms and the general term of the Maclaurin series for $\int_0^x f(t)\,dt$. In part (d) the student finds the appropriate approximation. The student should have used an approximation symbol, but the inappropriate use of the equal sign did not lose the first point. The sum does not need to be simplified. The only property of the terms that the student gives is that the series is alternating, and this is not enough to conclude that the approximation is within the given accuracy. The student earned the third point for establishing an error bound that is less than $\frac{1}{10{,}000}$.

Sample: 6B
Score: 6

This student earned 6 points: 1 point in part (a), 2 points in part (b), 1 point in part (c), and 2 points in part (d). In part (a) the first three nonzero terms of the Maclaurin series for $f'(x)$ are correct, but the general term is not provided. The computation of $f'(x)$ and evaluation of $f'\left(\frac{1}{2}\right)$ are correct, which earned all points in part (b). The first four nonzero terms in the Maclaurin series for $\int_0^x f(t)\,dt$ are correct in part (c), but the general term is not provided. For part (d) the first three nonzero terms of the Maclaurin series for $\int_0^{1/2} f(t)\,dt$ are correctly summed,

Question 6 (continued)

and the absolute value of the fourth nonzero term is shown to be smaller than $\frac{1}{10,000}$. However, there is no mention of the properties of the terms of the series that support the use of this error bound.

Sample: 6C
Score: 3

This student earned 3 points: 2 points in part (a) and 1 point in part (c). The work in part (a) is correct. In part (b) the student recognizes that the series is the value of the derivative at $x = \frac{1}{2}$ but attempts to approximate this using the first three terms of the series rather than compute the derivative directly and evaluate it at $x = \frac{1}{2}$. In part (c) the student makes an antidifferentiation error in the fourth term and did not earn the first point. The general term is correct, however, and earned the second point. In part (d) the student never finds the approximation using the first three terms. The student cites two properties of the terms but fails to include the property that the terms converge to zero and thus did not earn the second point. The reported error bound is actually greater than $\frac{1}{10,000}$, so the student did not earn the last point.